Thermodynamics in Quasi-Spherical Szekeres Space-Time

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We have considered that the universe is the inhomogeneous $(n+2)$ dimensional quasi-spherical Szekeres space-time model. We consider the universe as a thermodynamical system with the horizon surface as a boundary of the system. To study the generalized second law (GSL) of thermodynamics through the universe, we have assumed the trapped surface is the apparent horizon. Next we have examined the validity of the generalized second law of thermodynamics (GSL) on the apparent horizon by two approaches: (i) using first law of thermodynamics on the apparent horizon and (ii) without using the first law. In the first approach, the horizon entropy have been calculated by the first law. In the second approach, first we have calculated the surface gravity and temperature on the apparent horizon and then horizon entropy have found from area formula. The variation of internal entropy have been found by Gibb's law. Using these two approaches separately, we find the conditions for validity of GSL in $(n+2)$ dimensional quasi-spherical Szekeres model.

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I. INTRODUCTION

In Einstein gravity, the evidence of the connection between gravity and thermodynamics was first discovered in [1] by deriving the Einstein equation from the proportionality of entropy and horizon area together with the first law of thermodynamics. The horizon area of black hole is associated with its entropy, the surface gravity is related with its temperature in black hole thermodynamics [2]. Then Padmanabhan [3] was able to formulate the first law of thermodynamics on the horizon, starting from Einstein equations for a general static spherically symmetric space time. Frolov et al [4] calculated the energy flux of a background slow-roll scalar field through the quasi-de Sitter apparent horizon and used the first law of thermodynamics $-dE = TdS$, where dE is the amount of the energy flow through the apparent horizon. Using the Hawking temperature $T_A = \frac{1}{2\pi R_A}$ and Bekenstein entropy $S_A = \frac{\pi R_A^2}{G}$ (R_A is the radius of apparent horizon) at the apparent horizon, the first law of thermodynamics (on the apparent horizon) is shown to be equivalent to Friedmann equations [5] and the generalized second law of thermodynamics is obeyed at the horizon. The thermodynamics in de Sitter spacetime was first investigated by Gibbons and Hawking in [6]. When the apparent horizon and the event horizon of the Universe are different, it was found that the first law and generalized second law (GSL) of thermodynamics hold on the apparent horizon, while they break down if one considers the event horizon [7]. On the basis of the well known correspondence between the Friedmann equation and the first law of thermodynamics of the apparent horizon, Gong et al [8] argued that the apparent horizon is the physical horizon in dealing with thermodynamics problems. Considering FRW model of the universe, most studies deal with validity of the generalized second law of thermodynamics starting from the first law when universe is bounded by the apparent horizon [9]. But there are few works of the justification of first and second laws of thermodynamics on the event horizon [10]. The validity of thermodynamical laws in generalized gravity theories have also discussed in [11].

Usually, for cosmological phenomena over galactic scale or in the smaller scale, it is reasonable to consider inhomogeneous solutions to Einstein equations. Szekeres' [12] in 1975, gave a class of inhomogeneous solutions representing irrotational dust. The space-time represented by these solutions has no killing vectors and it has invariant family of spherical hypersurfaces. Hence this space-time is referred as quasi-spherical space-time. Subsequently, the solutions have been extended by Szafron [13] and Szafron and Wainwright [14] for perfect fluid and they studied asymptotic behaviour for different choice of the parameters involved. Later Barrow and Stein-Schabes [15] gave solutions for dust model with a cosmological constant and showed the validity of the

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Cosmic 'no-hair' Conjecture. Recently, Chakraborty et al [16] have extended the Szekeres solution to $(n+2)$ dimensional space-time and generalized it for matter containing heat flux [17]. Recently, several works have been done on gravitational collapse using this higher dimensional Szekeres solution [18,19].

In this work, we consider the $(n+2)$ dimensional quasi-spherical Szekeres space-time. Next we'll examine the validity of the generalized second law of thermodynamics (GSL) on the apparent horizon by two approaches: (i) using first law of thermodynamics on the apparent horizon and (ii) without using first law. In the first approach, we don't need the horizon temperature. So the horizon entropy can be calculated from the first law. In the second approach, first we calculate surface gravity and temperature on the apparent horizon and then horizon entropy can be found from area formula. Using these two approaches, we find the conditions for validity of GSL in quasi-spherical Szekeres model.

II. THE SZEKERES' MODEL

The metric ansatz for the $(n+2)$ dimensional Szekeres' space-time [12, 16] is of the form

$$ds^2 = -dt^2 + e^{2\alpha}dr^2 + e^{2\beta}\sum_{i=1}^{n} dx_i^2 \tag{1}$$

where the metric coefficients α and β are functions of all space-time co-ordinates i.e.,

$$\alpha = \alpha(t, r, x_1, ..., x_n), \quad \beta = \beta(t, r, x_1, ..., x_n).$$

Now Considering both radial and transverse stresses the energy momentum tensor has the structure

$$T_\mu^\nu = \text{diag}(-\rho, p_r, p_{_T}, ..., p_{_T})$$

Now for the choice namely $\beta' \neq 0$, $\dot{\beta}_{x_i} = 0$ we have from the field equations the explicit form of the metric coefficients are as follows [16]:

$$e^\beta = R(t,r)\ e^{\nu(r,x_1,...,x_n)} \tag{2}$$

and

$$e^\alpha = \frac{R' + R\ \nu'}{\sqrt{1+f(r)}} \tag{3}$$

and the evolution equation for R gives

$$R\ddot{R} + \frac{1}{2}(n-1)\dot{R}^2 + \frac{p_r}{n}R^2 = \frac{n-1}{2}\ f(r), \tag{4}$$

where $f(r)$ is the function of r. Also the function ν satisfies

$$e^{-2\nu}\sum_{i=1}^{n}[(n-2)\nu_{x_i}^2 + 2\nu_{x_i x_i}] = -n \tag{5}$$

which has a solution of the form

$$e^{-\nu} = A(r)\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r)x_i + C(r) \tag{6}$$

with the restriction,

$$\sum_{i=1}^{n} B_i^2 - 4AC = -1 \tag{7}$$

for the arbitrary functions $A(r)$, $B_i(r), (i = 1, 2, .., n)$ and $C(r)$.

Now from conservation equation $T^{\nu}_{\mu;\ \nu} = 0$ we get [19]

$$\left.\begin{aligned} \dot{\rho} + \dot{\alpha}(\rho + p_r) + n\dot{\beta}(\rho + p_{_T}) = 0 \\ p_r' + n\beta'(p_r - p_{_T}) = 0 \\ \text{and} \quad \alpha_{x_i}(p_r - p_{_T}) = \tfrac{\partial}{\partial x_i} p_{_T} \qquad (i = 1, 2, ..., n) \end{aligned}\right\} \tag{8}$$

In the general case when both radial and tangential pressures are non-zero and distinct then from the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ (choosing $8\pi G = c = 1$) they can be obtained in compact form as [19]

$$\rho = \frac{F'}{\zeta^n \zeta'} \quad , \quad p_r = -\frac{\dot{F}}{\zeta^n \dot{\zeta}} \quad \text{and} \quad p_{_T} = p_r + \frac{\zeta p_r'}{n\zeta'} \tag{9}$$

where

$$\zeta = e^{\beta} \quad \text{and} \quad F(t, r) = \frac{n}{2} R^{n-1} e^{(n+1)\nu}(\dot{R}^2 - f(r)) \tag{10}$$

Now consider radial and tangential pressures are equal i.e., $p_r = p_{_T} = p$, so from (8), we get the isotropic pressure is function of t only i.e., $p = p(t)$. As there is no restriction on the energy density so ρ is in general a function of all the $(n + 2)$ variables i.e., $\rho = \rho(t, r, x_1, ..., x_n)$ and hence no equation of state is imposed. So the conservation equation (8) yields to

$$\dot{\rho} + (\dot{\alpha} + n\dot{\beta})(\rho + p) = 0 \tag{11}$$

Also the metric (1) can be written as

$$ds^2 = -dt^2 + \frac{(R' + R\nu')^2}{1 + f(r)} dr^2 + R^2 e^{2\nu} \sum_{i=1}^{n} dx_i^2 \tag{12}$$

Here R is the radius of the non-concentric spheres. If the spheres are concentric i.e, if $\nu' = 0$ then the above Szekeres metric reduces to $(n+2)$ dimensional spherically symmetric Lamaître-Tolman-Bondi (LTB) metric [16].

III. GSL OF THERMODYNAMICS ON THE APPARENT HORIZON

Now we consider the metric (8) in the following form

$$ds^2 = h_{ab} dx^a dx^b + R^2 e^{2\nu} \sum_{i=1}^{n} dx_i^2 \quad , \quad a, b = 0, 1 \tag{13}$$

where $h_{ab} = diag\left(-1, \frac{(R'+R\nu')^2}{1+f(r)}\right)$.

The formation of event horizon depends greatly on the computation of null geodesics whose computation are almost impracticable for the present space-time geometry. So a closely related concept of a trapped surface (a space-like 2-surface whose normals on both sides are future pointing converging null geodesic families) will be considered. The dynamical apparent horizon R_A, a marginally trapped surface with vanishing expansion, is determined by the relation [5,19,21] (see also **APPENDIX**)

$$h^{ab}\partial_a(Re^{\nu})\partial_b(Re^{\nu}) = 0 \tag{14}$$

This implies

$$\dot{R}_A^2 = 1 + f(r) \tag{15}$$

So from equation (10), we obtain (on the apparent horizon)

$$F(t,r) = \frac{n}{2}R_A^{n-1}e^{(n+1)\nu} \tag{16}$$

Now the Gibb's law of thermodynamics states that [7]

$$T_A dS_I = pdV + d(E_I) \tag{17}$$

where, S_I, p, V and E_I are respectively entropy, pressure, volume and internal energy within the apparent horizon. Here the expression for internal energy can be written as $E_I = \rho V$. Here T_A is the temperature on the apparent horizon. Now the volume of the $(n+1)$ dimensional space is [5]

$$V = \Omega_{n+1}R_A^{n+1}e^{(n+1)\nu} \quad \text{where} \quad \Omega_{n+1} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})} \tag{18}$$

The time variation of internal entropy is obtained as (using (2), (3), (11), (17) and (18))

$$\dot{S}_I = \frac{\Omega_{n+1}R_A^{n+1}e^{(n+1)\nu}}{T_A}\,(\rho+p)\left(\frac{\dot{R}_A}{R_A} - \frac{\dot{R}'_A + \dot{R}_A\nu'}{R'_A + R_A\nu'}\right) \tag{19}$$

A. Validity Conditions of GSL using First Law of Thermodynamics

The unified first law is defined by [22]

$$dE = \mathcal{A}\Psi + WdV \tag{20}$$

where

$$\mathcal{A} = (n+1)\Omega_{n+1}R^n e^{n\nu} \tag{21}$$

is the area [5] and the volume V is defined in (18). The work density function is given by

$$W = -\frac{1}{2}h^{ab}T_{ab} = \frac{1}{2}(\rho - p) \tag{22}$$

The energy-supply vector is given by

$$\Psi_a = h^{bc}T_{ac}\partial_b(Re^{\nu}) + W\partial_a(Re^{\nu}) = \left(-\frac{1}{2}(\rho+p)\dot{R}e^{\nu}, \frac{1}{2}(\rho+p)(R' + R\nu')e^{\nu}\right) \tag{23}$$

So

$$\Psi = \Psi_a dx^a = -\frac{1}{2}(\rho+p)e^{\nu}[\dot{R}dt - (R' + R\nu')dr] \tag{24}$$

The total energy inside the quasi-spherical surface is given by

$$E = \frac{n(n+1)}{2}\Omega_{n+1}R^{n-1}e^{(n-1)\nu}[e^{2\nu} - h^{ab}\partial_a(Re^{\nu})\partial_b(Re^{\nu})] = \frac{n(n+1)}{2}\Omega_{n+1}R^{n-1}e^{(n+1)\nu}[\dot{R}^2 - f(r)] \tag{25}$$

Comparing (10) and (25), we get

$$E = (n+1)\Omega_{n+1}F \tag{26}$$

From this, we can say that $F(t,r)$ represents the mass function within the quasi-spherical surface. Now using (18), (21), (22) and (24), we get

$$A\Psi + WdV = (n+1)\Omega_{n+1}R^n e^{(n+1)\nu}[-p\dot{R}dt + \rho(R' + R\nu')dr] \tag{27}$$

Using (25) and (27), comparing the coefficients of dt and dr in (20), we can recover the field equations (4) and (9). Now from the unified first law (20) and using (27), we get

$$dE = (n+1)\Omega_{n+1}R^n e^{(n+1)\nu}[-(\rho+p)\dot{R}dt + \rho e^{-\nu}d(Re^{\nu})] \tag{28}$$

We know that heat is one of the forms of energy. Therefore, the heat flow δQ through the apparent horizon is just the amount of energy crossing it during the time interval dt. That is, $\delta Q = -dE$ is the change of the energy inside the apparent horizon. So the amount of the energy crossing on the apparent horizon is given by [20]

$$-dE_A = (n+1)\Omega_{n+1}R_A^n\dot{R}_A e^{(n+1)\nu}(\rho+p)dt = \mathcal{A}\dot{R}_A e^{\nu}T_{\mu\nu}k^{\mu}k^{\nu}dt \tag{29}$$

The first law of thermodynamics (Clausius relation) on the apparent horizon is defined as follows:

$$T_A dS_A = dQ = -dE_A \tag{30}$$

So using (29) and (30), we obtain the time variation of the entropy on the apparent horizon as

$$\dot{S}_A = \frac{(n+1)\Omega_{n+1}}{T_A}\, R_A^n\dot{R}_A e^{(n+1)\nu}(\rho+p) \tag{31}$$

Combining (19) and (31), we obtain

$$\dot{S}_I + \dot{S}_A = \frac{\Omega_{n+1}R_A^{n+1}e^{(n+1)\nu}}{T_A}\,(\rho+p)\left((n+2)\frac{\dot{R}_A}{R_A} - \frac{\dot{R}'_A + \dot{R}_A\nu'}{R'_A + R_A\nu'}\right) \tag{32}$$

Using (2), (3), (9), (15), (16) and (32), after manipulation we get the rate of change of total entropy as

$$\dot{S}_I + \dot{S}_A = \frac{\Omega_{n+1}F}{T_A}\left[\left(\left(\frac{n}{2}\right)^{\frac{1}{n-1}}\frac{(n-1)F'\sqrt{1+f}}{F' - 2F\nu'}F^{\frac{n-2}{n-1}}e^{\frac{n+1}{n-1}\nu} - \dot{F}\right)\times\right.$$
$$\left.\left(\frac{(n+2)}{F} - \frac{(n-1)(f' + 2(1+f)\nu')}{2(1+f)\,(F' - 2F\nu')}\right)\right] \tag{33}$$

If the expression inside the square bracket is non-negative then the GSL will be justified. For marginally bound case, i.e., for $f(r)=0$, the GSL is satisfied if the following conditions hold:

(i) $F' \geq \frac{3(n+1)}{n+2}F\nu'$ and $\dot{F} \leq 3(n+1)\left(\frac{n}{2}\right)^{\frac{1}{n-1}}F^{\frac{n-2}{n-1}}e^{\frac{n+1}{n-1}\nu}$

OR, (ii) $F' \leq \frac{3(n+1)}{n+2}F\nu'$ and $\dot{F} \geq 3(n+1)\left(\frac{n}{2}\right)^{\frac{1}{n-1}}F^{\frac{n-2}{n-1}}e^{\frac{n+1}{n-1}\nu}$.

B. Validity Conditions of GSL without using First Law of Thermodynamics

The surface gravity is defined as

$$\kappa=\frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b(Re^{\nu})) \tag{34}$$

where $h=\det(h_{ab})$. So on the apparent horizon, we get

$$\kappa=-\frac{\ddot{R}_A e^{\nu}}{2}-\frac{\dot{R}_A(\dot{R}'_A+\dot{R}_A\nu')e^{\nu}}{2(R'+R\nu')}+\frac{\sqrt{1+f}}{2(R'+R\nu')}\frac{\partial}{\partial r}(e^{\nu}\sqrt{1+f}) \tag{35}$$

Now apparent horizon temperature is (using (3)-(5), (9), (10), (15) and (16))

$$T_A=\frac{|\kappa|}{2\pi}=\frac{e^{\nu}}{8\pi}\left|(n-1)\left(\frac{n}{2F}\right)^{\frac{1}{n-1}}e^{\frac{n+1}{n-1}\nu}-\frac{\dot{F}}{F\sqrt{1+f}}\right| \tag{36}$$

Since one can relate the entropy with the surface area of the apparent horizon through $S_A=\mathcal{A}/4G$. Therefore using (21) we have

$$S_A=2\pi(n+1)\Omega_{n+1}R_A^n e^{n\nu}\quad,\qquad\qquad(\text{since}\ \ 8\pi G=1) \tag{37}$$

So the variation of entropy on the apparent horizon is obtained as (using (15) and (16))

$$\dot{S}_A=4\pi(n+1)\Omega_{n+1}F\sqrt{1+f}\ e^{-\nu} \tag{38}$$

Using (9), (10), (15), (16), (19), (36) and (38), we finally obtain (after manipulation) the rate of change of total entropy as

$$\dot{S}_I+\dot{S}_A=\frac{\Omega_{n+1}F}{T_A}\left[\left(\left(\frac{n}{2}\right)^{\frac{1}{n-1}}\frac{(n-1)F'\sqrt{1+f}}{F'-2F\nu'}F^{\frac{n-2}{n-1}}e^{\frac{n+1}{n-1}\nu}-\dot{F}\right)\times\right.$$

$$\left.\left(\frac{1}{F}-\frac{(n-1)(f'+2(1+f)\nu')}{2(1+f)\ (F'-2F\nu')}\right)+\frac{(n+1)\sqrt{1+f}}{2}\times\left|(n-1)\left(\frac{n}{2F}\right)^{\frac{1}{n-1}}e^{\frac{n+1}{n-1}\nu}-\frac{\dot{F}}{F\sqrt{1+f}}\right|\ \right] \tag{39}$$

If the expression inside the square bracket is non-negative then the GSL will be justified. From the above expression, we cannot draw any definite conclusion.

IV. DISCUSSIONS

We have considered that the universe is the inhomogeneous $(n+2)$ dimensional quasi-spherical Szekeres space-time model. We consider the universe as a thermodynamical system with the horizon surface as a boundary of the system. To study the generalized second law (GSL) of thermodynamics through the universe, we have assumed the trapped surface is the apparent horizon. Next we have examined the validity of the generalized second law of thermodynamics (GSL) on the apparent horizon by two approaches: (i) using first law of thermodynamics on the apparent horizon and (ii) without using the first law. In the first approach, the horizon entropy have been calculated by the first law. In the second approach, first we have calculated the surface gravity and temperature on the apparent horizon and then horizon entropy have found from area formula. The variation of internal entropy have been found by Gibb's law. Using these two

approaches separately, we find the conditions for validity of GSL in $(n+2)$ dimensional quasi-spherical Szekeres model. Also for marginally bound case, we have found the bounds on the derivatives of the mass function F.

APPENDIX

Define, $X = \frac{R'+R\nu'}{\sqrt{1+f(r)}}$ and $Y = Re^{\nu}$. Let us consider the 2-surface $S_{r,t}$ (r =constant, t =constant) is a trapped surface and K^{μ} denotes the tangent vector field to the null geodesics which is normal to $S_{r,t}$. So on the apparent horizon (on $S_{r,t}$) we have [21]

$$K_{\mu}\ K^{\mu} = 0, \quad K^{\mu}_{\ ;\ \nu}\ K^{\nu} = 0 \tag{40}$$

and

$$K^2 = K^3 = 0, \quad (K^0)^2 - X^2(K^1)^2 = 0 \tag{41}$$

Now on $S_{r,t}$, the choice of affine parameter may clearly be such that

$$K^0 = X, \ K^1 = \epsilon = \pm 1 \tag{42}$$

Now on $S_{r,t}$,

$$K^{\mu}_{;\mu} = K^{\mu}_{,\mu} + \Gamma^{\mu}_{\mu\nu}K^{\nu} = K^0_{,0} + K^1_{,1} + X\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right) + \epsilon\left(\frac{X'}{X} + \frac{Y'}{Y}\right) \tag{43}$$

Since $K^2_{,2} = K^3_{,3} = 0$ on $S_{r,t}$. On the other hand, forming $\partial/\partial t$ of the first equation of (32) and setting $\mu = 1$ in the second equation gives on $S_{r,t}$

$$K^0_{,0} - \epsilon X K^1_{,0} - \dot{X} = 0 \tag{44}$$

and

$$K^1_{,0}X + \epsilon(K^1_{,1} + 2\dot{X}) + \frac{X'}{X} = 0 \tag{45}$$

Eliminating K^1_0 between these two equations and substituting in (35) gives

$$K^{\mu}_{;\mu} = \frac{2}{Y}(X\dot{Y} + \epsilon Y') = \frac{2(R'+R\nu')}{R}\left(\frac{\dot{R}}{\sqrt{1+f(r)}} + \epsilon\right) \tag{46}$$

On apparent horizon, $K^{\mu}_{;\mu} = 0$ gives

$$\frac{\dot{R}}{\sqrt{1+f(r)}} + \epsilon = 0 \qquad (\text{since}, \quad R' + R\nu' \neq 0) \tag{47}$$

$$\Rightarrow \quad \dot{R}^2 = \epsilon^2(1+f(r)) = 1 + f(r) \tag{48}$$

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